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THE CURVATURE OF

PLANE ELASTIC CURVES

by

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The Curvature of Plane Elastic Curves

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Abstract

In this paper plane elastic curves are revisited from a viewpoint that emphasizes curvature properties of these curves. The family of elastic curves is considered in dependence of a tension parameter σ and the squared global curvature maximum κ_m^2. It is shown that for any elastic curve κ_m^2 is bigger than the tension parameter σ. A curvature analysis of the fundamental forms of the elastic curves is presented. A formula is established that gives the maximum turning angle of an elastica as a function depending on κ_m^2 and σ. Finally, it is shown that an elastic curve can be represented as a linear combination of its curvature, arc length and energy function and that any curve with this property is an elastica.

1 Introduction

The search for Smoothing Algorithms in CAGD has induced research on curves and surfaces which minimize functionals with geometrical or physical meaning. The classical example are elastica which describe the shape of elastic materials. Plane elastica can be defined as the extremals (critical points) of the variational problem

$$\int_0^L \kappa^2(s) + \sigma \, ds \rightarrow \quad \min$$

where κ denotes the curvature, s the arc length of a plane curve x and σ is constant. The total length L of x is considered to be variable. The set of comparison curves of the problem is the set of all C^∞ curves in the plane with fixed endpoints and fixed tangent directions at these points.

Lee and Forsythe found (see [6]) that the curvature function of an elastica satisfies the differential equation

$$\kappa''(s) + \frac{1}{2}\kappa^3(s) + K\kappa(s) = 0, \quad K \in \mathbf{R}. \tag{1}$$

We begin this paper with a short derivation of this result and a proof that this equation is also sufficient for a curve to be an elastica. We will then show that for any elastica the square of the curvature function has a global maximum κ_m^2 which is bigger than the tension parameter σ. The fundamental forms of elastic curves first described by Euler are characterized by the values of the parameters κ_m^2 and σ. A curvature analysis of these curves is presented.

In section 3 various formulas expressing properties of elastica in terms of κ_m^2 and σ are given. This includes a formula that gives the maximum turning angle of an elastica as a function depending on κ_m^2 and σ. The explicit formula for the maximum turning angle allows to determine an upper bound for κ_m^2 in the case that an inflectional elastica has a turning angle bigger than π. As the main result of this section we show that a plane elastica has a representation as a linear combination of its curvature, arc length and energy function $E(s) = \int_0^s \kappa^2(\bar{s})\, d\bar{s}$. This property characterizes the class of plane elastica and can be used for speeding up the computation of interpolating elastica.

2 The curvature of plane elastica

Given two points $P, Q \in \mathbf{R}^2$ and two unit vectors $V \in T_P\mathbf{R}^2, W \in T_Q\mathbf{R}^2$ then M denotes the set

$$\begin{aligned} M := \{x : [0,L] \to \mathbf{R}^2 : \quad & L \in \mathbf{R}^+, x \in C^\infty[0,L], |x'(s)| = 1 \text{ for } s \in [0,L], \\ & x(0) = P,\ x(L) = Q,\ x'(0) = V,\ x'(L) = W\}. \end{aligned}$$

We consider the problem of minimizing the functional

$$E + \sigma L := \int_0^L \kappa^2(s) + \sigma\, ds$$

among all curves of M where κ denotes the curvature and s the arc length parameter of a plane curve x. The integral $E = \int_0^L \kappa^2(s)\, ds$ is called the energy of x on $[0, L]$. σ denotes a constant which we call the tension parameter. The total length L is considered to be variable. As the functional considered is translation and rotation invariant we assume in this section that $P = (0,0)$ and $V = (1,0)$.

For $x \in M$ the tangent vector T of x is given by

$$T(s) = (\cos(\Psi(s)), \sin(\Psi(s)))$$

where the function Ψ with $\Psi(s) := \int_0^s \kappa(\bar{s})\, d\bar{s}$ gives the turning angle of x. Using this notation the variational problem can be written as

$$\min_{\Psi \in C^\infty[0,L]} \int_0^L \Psi'^2(s) + \sigma \, ds$$

under the constraint

$$\int_0^L \begin{pmatrix} \cos \Psi(s) \\ \sin \Psi(s) \end{pmatrix} ds = Q,$$

where the admissible functions $\Psi \in C^\infty[0,L]$ are subject to the constraints:

$$\Psi(0) = 0, \quad \Psi(L) = \psi \quad \text{with} \quad (\cos\psi, \sin\psi) = W.$$

According to Lagrange's multiplier rule the differential equation of this problem is the Euler equation for the integrand

$$F(\Psi, \Psi', s) = \Psi'^2(s) + \sigma + \lambda \cos \Psi(s) + \mu \sin \Psi(s),$$

for some constants $\lambda, \mu \in \mathbf{R}$, i.e.

$$\Psi''(s) = -\frac{\lambda}{2} \sin \Psi(s) + \frac{\mu}{2} \cos \Psi(s). \tag{2}$$

(see e.g. in [3],[6]). By introducing the constants

$$\lambda = 2a \cos \phi \quad \mu = 2a \sin \phi$$

we rewrite the Euler equation in the form

$$\Psi'' = \kappa' = -a \sin(\Psi - \phi). \tag{3}$$

Multiplying (3) by $2\Psi'$ and integrating yields

$$\kappa^2 = 2a \cos(\Psi - \phi) + A \tag{4}$$

where $A \in \mathbf{R}$ denotes an integration constant. Equation (4) has been used to define elastic curves in the plane in the classic literature (see [7]).

In order to determine the constant A in terms of the tension parameter σ, we consider the boundary condition

$$F(\Psi(L), \Psi'(L), L) - \frac{\partial}{\partial \Psi'} F(\Psi(s), \Psi'(s), s)_{|L} \Psi'(L) = 0$$

that must be satisfied by the extremal. This condition is implied by the fact that the total length L of the curve is variable in the variation (see e.g. [2], p. 571). Thus,

$$\Psi'^2(L) = \kappa^2(L) = \lambda \cos\Psi(L) + \mu \sin\Psi(L) + \sigma. \tag{5}$$

Comparing (5) with (4) shows that

$$A = \sigma.$$

Therefore we give the following definition of an elastic curve under tension.

Definition 1 *An arc length parametrized plane curve x with curvature function κ is called elastica (or elastic curve) with tension parameter σ, if for some $a, \varphi \in \mathbf{R}$*

$$\kappa^2 = 2a\cos(\Psi - \phi) + \sigma \tag{6}$$

where Ψ denotes the function $\Psi(s) := \int_0^s \kappa(\bar{s})\, d\bar{s}$.

Lee and Forsythe showed in [6] that the Euler equation (2) implies the differential equation (1) for the curvature function κ. We will now show that (1) is in fact equivalent to (6).

Theorem 2 *If $\kappa \in C^2(\mathbf{R})$ and $\Psi(s) := \int_0^s \kappa(\bar{s})\, d\bar{s}$, then*

$$\kappa^2 = 2a\cos(\Psi - \phi) + \sigma$$

holds for some constants a, φ and σ if and only if

$$\kappa'' + \frac{1}{2}\kappa^3 - \frac{1}{2}\sigma\kappa = 0. \tag{7}$$

Proof: (i) Differentiating (6) yields the Euler equation in the form of (3). Differentiating the Euler equation gives

$$\kappa'' = -a\cos(\Psi - \phi)\kappa.$$

Substituting the term $\cos(\Psi - \varphi)$ in the above equation according to (6) we obtain (7).

(ii) Differential equation (7) can be integrated to the first oder differential equation

$$(\kappa')^2 = C - (1/4)\sigma^2 - (1/4)(\kappa^2 - \sigma)^2. \tag{8}$$

Note, that for a real solution of (8) it is necessary that

$$a^2 := C - (1/4)\sigma^2 > 0$$

and

$$\frac{(\kappa^2-\sigma)^2}{4a^2} \leq 1.$$

The function θ defined by

$$\theta := (-1)^n \arccos(\frac{\kappa^2-\sigma}{2|a|})$$

with $n=1$ for $\kappa' > 0$ and $n=2$ for $\kappa' \leq 0$ obviously obeys the relation

$$\kappa^2 = 2|a|\cos\theta + \sigma. \tag{9}$$

It remains to show that θ is an integral of κ.

(8) together with the definition of a^2 and (9) implies

$$(\kappa')^2 = a^2 \sin^2\theta.$$

Since $\sin\theta$ is positive (negative) if κ' is negative (positive), we obtain

$$\kappa' = -|a|\sin\theta. \tag{10}$$

Differentiating (9) and substituting κ' according to (10) yields

$$\theta' = \kappa.$$

Since Ψ and θ are both integrals of κ there is a constant ϕ such that $\Psi = \theta + \phi$. For Ψ

$$\kappa^2 = 2|a|\cos(\Psi - \phi) + \sigma$$

holds because of (9).

□

The squared curvature function κ^2 of an elastic curve has a global maximum, even if κ^2 is extended to the whole real line.

Lemma 3 *If κ is a solution of (7) on $\mathbf{R}$, then κ^2 has a global maximum.*

Proof: Since the function $(1/2)(\kappa^3 - \sigma\kappa)$ is of the class $C^1(\mathbf{R})$ the solutions of (7) can be extended to the whole real line.

To show that κ^2 has a global maximum we observe first that (3) implies the existence of a local extremum of κ. This is because the assumption $\kappa'(s) = \Psi''(s) \neq 0$ for all $s \in \mathbf{R}$ means that Ψ is convex or concave and therefore unbounded while according to (6) κ' has no zeros only if Ψ is bounded. Furthermore from (3) and (4) it is obvious that any local extremum of κ is a global extremum of κ^2.

We assume now that κ^2 has no global maximum on $\mathbf{R}$. In this situation κ' has exactly one zero s_{min} on $\mathbf{R}$ and κ^2 takes its global minimum in this point. Note that κ is non-zero for any $s \neq s_{min}$ because a zero of κ at a point $\bar{s} \neq s_{min}$ would imply that $\kappa(s_{min}) = 0$ and $\kappa'(\tilde{s}) = 0$ for some point $\tilde{s}$ between $\bar{s}$ and s_{min}. Therefore $(\kappa^2)'$ has no zero besides s_{min} and κ^2 is strictly monotone increasing on the right of s_{min}. Hence κ is monotone increasing resp. decreasing on the right of s_{min} if κ has positive resp. negative values on the right of s_{min}. The formula

$$\Psi(s) = \int_{s_{min}}^{s} \kappa(\bar{s})\, d\bar{s} + \Psi(s_{min})$$

yields that Ψ is in any case unbounded. This is a contradiction to the assumption because (4) implies the existence of global maxima for κ^2 if Ψ is unbounded.

□

We now give the relation between the global maximum κ_m^2 of κ^2 and the tension parameter σ and express κ in terms of elliptic functions.

Theorem 4 *Let $\kappa \in C^2(\mathbf{R})$ be a solution of the differential equation (7) with a global maximum $\kappa_m^2 \neq 0$ of κ^2 on $\mathbf{R}$. Then the following statements hold:*
(i) $\kappa_m^2 \geq \sigma$
(ii) κ has a zero if and only if $\kappa_m^2 > 2\sigma$. In this case κ is given by

$$\kappa(s) = \kappa_m \operatorname{cn}(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \,|\, k^2) \tag{11}$$

with the parameter

$$k^2 = \frac{\kappa_m^2}{2(\kappa_m^2 - \sigma)}.$$

(iii) $2\sigma - \kappa_m^2 \leq \kappa^2 \leq \kappa_m^2$ for $\kappa_m^2 \leq 2\sigma$. In this case κ is given by

$$\kappa(s) = \kappa_m \operatorname{dn}(\kappa_m(s - s_m)/2 \,|\, \frac{1}{k^2}). \tag{12}$$

Proof: As a global maximum point s_m of κ^2 is also a zero of κ' (8) implies

$$C = \frac{1}{4}\kappa_m^4 - \frac{1}{2}\sigma\kappa_m^2.$$

Hence (8) takes the form

$$(\kappa')^2 = \frac{1}{4}(\kappa_m^2 - \kappa^2)(\kappa^2 + \kappa_m^2 - 2\sigma). \tag{13}$$

As all quantities in (13) are real (i) follows from the fact that the term $(\kappa_m^2 - \kappa^2)$ is always non-negative and therefore the term $(\kappa^2 + \kappa_m^2 - 2\sigma)$ has also to be non-negative for any κ^2.

Using the same argument we observe that $\kappa^2(s) = 0$ for some s implies that $\kappa_m^2 \geq 2\sigma$ while for $\kappa_m^2 \leq 2\sigma$ κ^2 has to be greater or equal $2\sigma - \kappa_m^2$.

To express κ in terms of elliptic functions we procede as follows. In the case that $\kappa_m^2 \geq 2\sigma$ we substitute $z^2 = (\kappa_m^2 - \kappa^2)/\kappa_m^2$ in (13) and obtain for z the differential equation

$$(z')^2 = \frac{1}{2}(\kappa_m^2 - \sigma)(1 - z^2)(1 - k^2 z^2) \tag{14}$$

where

$$k^2 = \frac{\kappa_m^2}{2(\kappa_m^2 - \sigma)}.$$

(14) is the differential equation of Jacobi's function sn=sn(u) for the argument $u = \sqrt{\frac{1}{2}(\kappa_m^2 - \sigma)}s$ (see [1],p. 114). We therefore obtain

$$z^2(s) = \mathrm{sn}^2(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \,|\, k^2).$$

The relation $\kappa^2 = \kappa_m^2(1 - z^2)$ then gives

$$\kappa^2(s) = \kappa_m^2 \mathrm{cn}^2(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \,|\, k^2) \tag{15}$$

(see [8],p.16). Since κ is differentiable, (15) implies (11).

In the case that $\kappa_m^2 \leq 2\sigma$ we substitute $z^2 = (\kappa_m^2 - \kappa^2)/2(\kappa_m^2 - \sigma)$ in (14) and obtain for z the differential equation

$$(z')^2 = \frac{1}{4}\kappa_m^2(1 - z^2)(1 - l^2 z^2)$$

where

$$l^2 = \frac{1}{k^2} = \frac{2(\kappa_m^2 - \sigma)}{\kappa_m^2}.$$

Hence

$$z^2(s) = \mathrm{sn}^2(\kappa_m(s - s_m)/2 \,|\, l^2).$$

The relation

$$\kappa^2 = \kappa_m^2(1 - l^2 z^2)$$

yields

$$\kappa^2(s) = \kappa_m^2 \mathrm{dn}^2(\kappa_m(s - s_m)/2 \,|\, l^2)$$

(see [8],p.16) which implies (13) again because of the differentiability of κ.

□

Theorem 2 implies that the curvature function of an elastic curve extends to a periodic function on $\mathbf{R}$. In the case that $\kappa_m^2 > 2\sigma$ this periodic function is symmetric with respect to any zero of its derivative and antisymmetric with respect to any zero. According to Love [7] this situation is called **inflectional** because the extension of x has turning points. In the case that $\kappa_m^2 \leq 2\sigma$ the extension of κ has no zeros but is still symmetric with respect to any zero of its derivative. The local extrema in this case are κ_m and $\pm(2\sigma - \kappa_m^2)$. The situation is illustrated in figure 1 - figure 8 where for $\kappa_m = 1$ various curvature functions and the corresponding elastica are shown. If only non-negative tension values are considered σ is an element of $[0, 1]$. With σ increasing from 0 to 1 the curvature function changes continuously from a lemniscate function ($\sigma = 0$) to a constant ($\sigma = 1$). A classification of the different forms of elastica has been given by Euler (see [5], [7]).

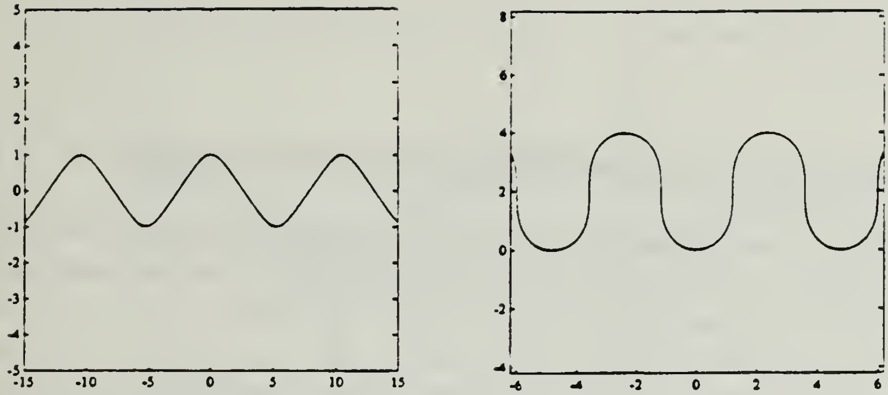

Figure 1. $\sigma = 0$; $\kappa(s) = \kappa_m \operatorname{coslemn}(\kappa_m(s - s_m)/2)$.

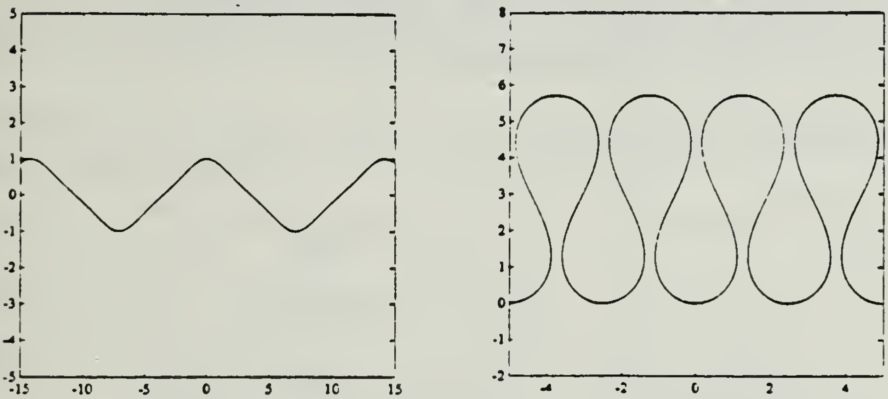

Figure 2. $\sigma = 0.3$; $\kappa(s) = \kappa_m \operatorname{cn}(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \,|\, k^2)$.

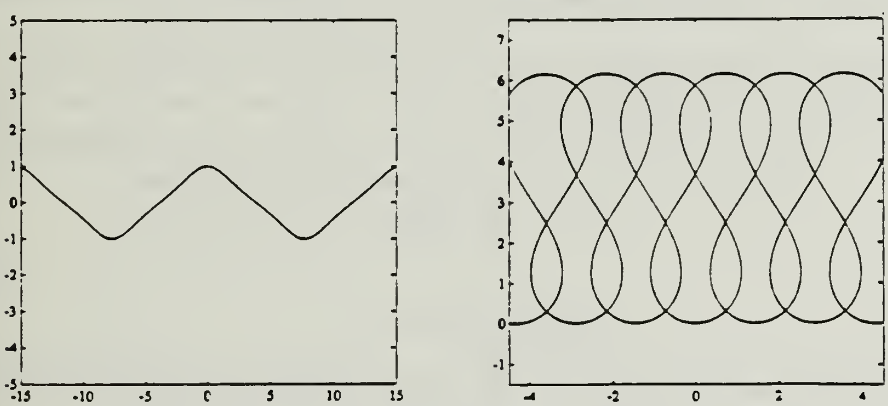

Figure 3. $\sigma = 0.35$; $\kappa(s)$ as in figure 2.

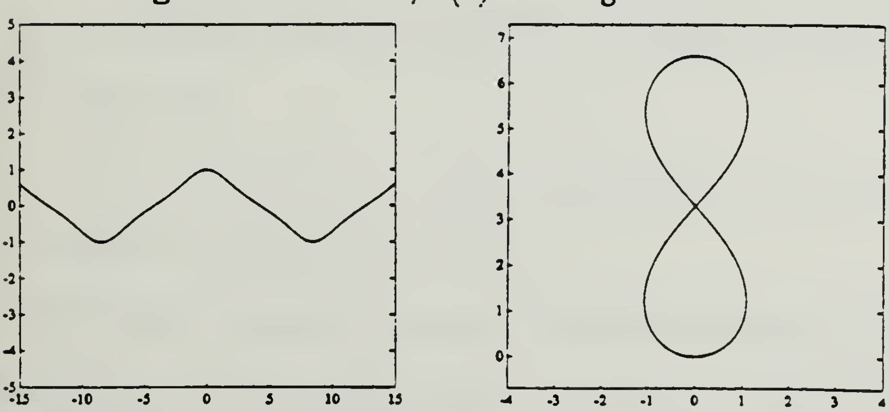

Figure 4. $\sigma = 0.394757217$; $\kappa(s)$ as in figure 3.

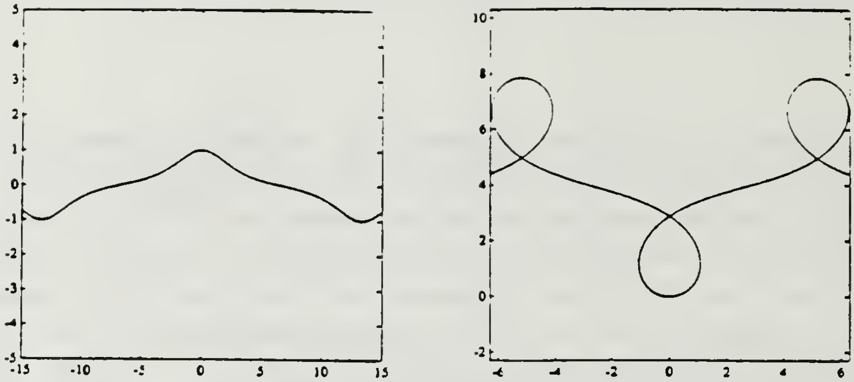

Figure 3. $\sigma = 0.49$; $\kappa(s) = \kappa_m \,\mathrm{cn}(\sqrt{(\kappa_m^2 - \sigma)/2}(s - s_m) \mid k^2)$.

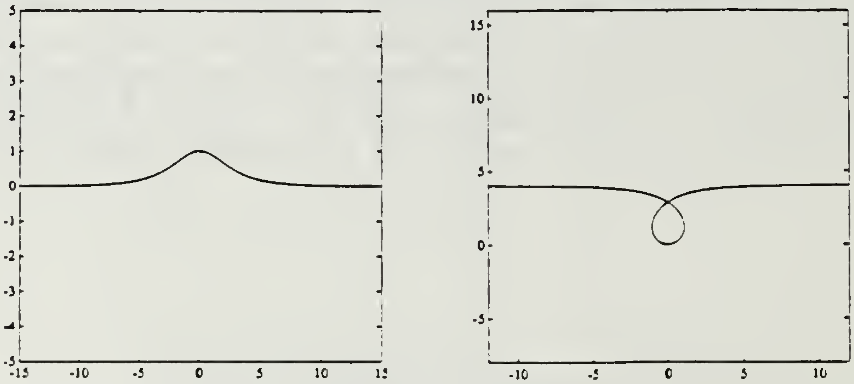

Figure 4. $\sigma = 0.5$; $\kappa(s) = \kappa_m \,\mathrm{sech}(\kappa_m(s - s_m)/2)$.

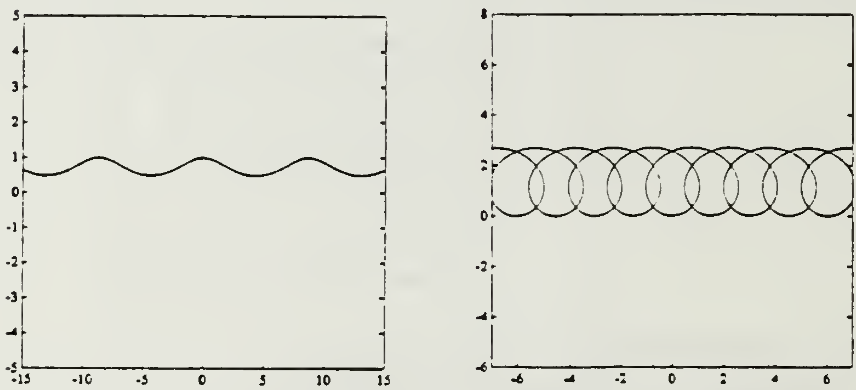

Figure 5. $\sigma = 0.62$; $\kappa(s) = \kappa_m \,\mathrm{dn}(\kappa_m(s - s_m)/2 \mid \frac{1}{k^2})$.

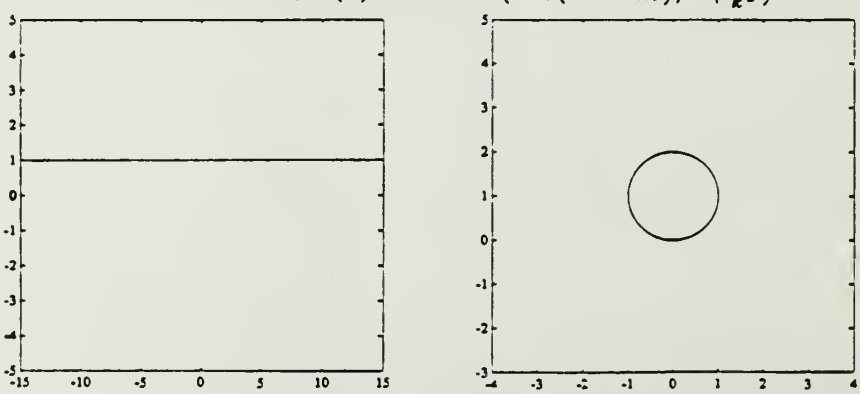

Figure 6. $\sigma = 1$; $\kappa(s) = \kappa_m$.

3 A characterizing representation theorem

First we summarize the main formulas for elastic curves. In order to avoid repetitions we use the following convention: if a curve $x : [0, L] \to \mathbf{R}^2$ is an elastica , then κ denotes the (analytic) curvature function of x, $\bar{\kappa}$ denotes the extension of κ on $\mathbf{R}$, κ_m^2 the global maximum of $\bar{\kappa}^2$ and s_m is a number with $\bar{\kappa}^2(s_m) = \kappa_m^2$. Furthermore $\Psi : [0, L] \to \mathbf{R}$ is the function $\Psi(s) := \int_0^s \kappa(\bar{s})\, d\bar{s}$, $\bar{\Psi}$ denotes the extension of Ψ on $\mathbf{R}$ and $\varphi := \bar{\Psi}(s_m)$. The energy E of x is the function $E(s) = \int_0^s \kappa^2(\bar{s})\, d\bar{s}$. For convenience we also use the notation $C(\psi) = (\cos\psi, \sin\psi)$. ϑ is the angle in $[0, 2\pi[$ with $x'(0) = C(\vartheta)$.

Theorem 5 *For a plane elastic curve x with tension parameter σ the following relations hold:*

(i) $\kappa'(s) = -\frac{1}{2}(\kappa_m^2 - \sigma)\sin(\Psi - \varphi)$,

(ii) $\kappa^2(s) = (\kappa_m^2 - \sigma)\cos(\Psi(s) - \varphi) + \sigma$,

(iii) $\kappa(s) - \kappa(0) = -\frac{1}{2}(\kappa_m^2 - \sigma) < C'(\varphi + \vartheta), x(s) - x(0) >$.

(iv) $E(s) = (\kappa_m^2 - \sigma) < C(\varphi + \vartheta), x(s) - x(0) > +\sigma s$,

Proof: Without loss of generality we can assume that a in (4) is positive. From lemma 3 we know that $\bar{\kappa}^2$ has a global maximum κ_m^2. Since for positive a the maximum occurs iff $\cos(\Psi(s_m) - \phi) = 1$, it follows that

$$2a = \kappa_m^2 - \sigma$$

and

$$\phi = \Psi(s_m) + 2k\pi.$$

Therefore (3) and (4) imply (i) and (ii). To obtain (iii) we rewrite (i) as

$$\kappa'(s) = -\frac{1}{2}(\kappa_m^2 - \sigma)(\sin(\Psi + \vartheta)\cos(\varphi + \vartheta) - \cos(\Psi + \vartheta)\sin(\varphi + \vartheta))$$

and integrate using

$$x'(s) = (\cos(\Psi(s)), \sin(\Psi(s))).$$

To verify (iv) we write (ii) in the form

$$\kappa^2(s) = (\kappa_m^2 - \sigma)(\cos(\Psi + \vartheta)\cos(\varphi + \vartheta) + \sin(\Psi + \vartheta)\sin(\varphi + \vartheta)) + \sigma$$

and integrate.

□

Since the elliptic function cn in Theorem 3 has the symmetry properties of a sine wave while dn is positive, the turning angle of an elastic curve is bounded in the inflectional case $\sigma \leq \frac{1}{2}\kappa_m^2$ and unbounded otherwise. Formula (ii) of Theorem 3 can be used to determine the maximum turning angle of an inflectional elastic curve.

Corollary 6 *The maximum turning angle $\Psi_{max} := \max_{s\in[t,t+T]} |\bar{\Psi}(s)|$ of an elastica x with period T and tension parameter $\sigma \leq \frac{1}{2}\kappa_m^2$ is given by*

$$\Psi_{max} = 2\arccos(\frac{\sigma}{\sigma - \kappa_m^2}). \tag{16}$$

If $\sigma > \frac{1}{2}\kappa_m^2 \neq 0$ then

$$|\bar{\Psi}(t+T) - \bar{\Psi}(t)| = 2\pi$$

for any $t \in \mathbf{R}$.

Proof: Let s_0 be a zero of $\bar{\kappa}$. Then it follows from the symmetry properties of $\bar{\kappa}$ in the inflectional case that

$$\Psi_{max} = 2|\int_{s_0}^{s_0+T/4} \bar{\kappa}(s)\,ds| = 2|\bar{\Psi}(s_0 + T/4) - \bar{\Psi}(a)|.$$

With (ii) we get

$$\Psi_{max} = 2|\arccos(\frac{\kappa_m^2 - \sigma}{\kappa_m^2 - \sigma}) + \varphi - \arccos(\frac{-\sigma}{\kappa_m^2 - \sigma}) - \varphi|$$

and therefore (16).

In the non-inflectional case κ is a positive periodic with the period $T = 4K/|\kappa_m|$ where K denotes the complete elliptic integral of the first kind. For $t = s_m$ one obtains

$$|\bar{\Psi}(s_m + T) - \bar{\Psi}(s_m)| = 4\int_0^K \mathrm{dn}(u|l^2)\,du = 4\arcsin(\mathrm{sn}(K)) = 2\pi.$$

□

For an inflectional elastica x formula (16) implies an upper bound for κ_m^2 if the absolute value of the oriented angle between $x'(0)$ and $x'(L)$ is bigger than π.

Corollary 7 *For an inflectional elastica x with tension parameter σ and $\psi := |\Psi(L)| > \pi$:*

$$2 \leq \frac{\kappa_m^2}{\sigma} \leq 1 - \frac{1}{\cos(\psi/2)}.$$

Proof: It follows from (16) that a turning angle bigger than π can only happen for positive σ. The inflectional nature of x implies therefore the left inequality. The right inequality follows from the fact that ψ has to be less or equal Ψ_{max} which is given by (16).

□

As the main result of this paragraph we show that an elastica can be represented as a linear combination of its curvature, energy and arc length.

Theorem 8 *If x is an elastica with tension parameter $\sigma \neq \kappa_m^2$ then*

$$x(s) = \frac{1}{\kappa_m^2 - \sigma} \begin{pmatrix} \sin(\varphi + \vartheta) & \cos(\varphi + \vartheta) \\ -\cos(\varphi + \vartheta) & \sin(\varphi + \vartheta) \end{pmatrix} \begin{pmatrix} 2(\kappa(s) - \kappa(0)) \\ E(s) - \sigma s \end{pmatrix} + x(0) \quad (17)$$

where $\cos\varphi = (\kappa_0^2 - \sigma)/(\kappa_m^2 - \sigma)$ *and* $\sin\varphi = 2\kappa_0'/(\kappa_m^2 - \sigma)$.

Proof: A plane curve $x \in C^\infty[0, L]$ has a representation

$$\begin{aligned} x(s) - x(0) &= \int_0^s C(\Psi(\bar{s}) - \varphi + \varphi + \vartheta)\, d\bar{s} \\ &= C(\varphi + \vartheta) \int_0^s \cos(\Psi(\bar{s}) - \varphi)\, d\bar{s} \\ &+ C'(\varphi + \vartheta) \int_0^s \sin(\Psi(\bar{s}) - \varphi)\, d\bar{s} \,. \end{aligned}$$

Applying (iii) and (iv) of Theorem 3 yields

$$x(s) - x(0) = \frac{1}{\kappa_m^2 - \sigma} \Big(C(\varphi + \vartheta)(E(s) - \sigma s) - C'(\varphi + \vartheta) 2(\kappa(s) - \kappa(0)) \Big)$$

which is equivalent to (17).

The formulas for φ follow from (i) and (ii) for $s = 0$.

□

Formula (17) provides an explicit representation of an elastica in terms of its curvature function if $\kappa_m^2 \neq \sigma$. Note that in the case $\kappa_m^2 = \sigma$ the elastic curve is a circle of radius $1/|\kappa_m|$ as shown in figure 8.

Finally we show that elastica are essentially the only curves in the plane that have a representation of the form (17).

Theorem 9 *Let κ be an arbitrary $C^2(\mathbf{R})$ function with a global maximum κ_m^2 of κ^2, σ a real number smaller than κ_m^2 and $E(s) := \int_0^s \kappa^2(\bar{s})\, d\bar{s}$. An arc length parametrized curve x where $x(s)$ is given by (17) is an elastica with curvature function κ and tension parameter σ.*

Proof: We assume the case $\varphi + \vartheta = \pi/2$ which can always be achieved by applying a rotation to x. Then

$$x' = \frac{1}{\kappa_m^2 - \sigma}(2\kappa', \kappa^2 - \sigma), \quad x'' = \frac{1}{\kappa_m^2 - \sigma}(2\kappa'', 2\kappa\kappa').$$

x is arc length parametrized if and only if $x' = 1$, i.e.

$$(2\kappa')^2 = (\kappa_m^2 - \sigma)^2 - (\kappa^2 - \sigma)^2. \tag{18}$$

Differentiating (18) yields

$$2\kappa'\kappa'' + \kappa'\kappa(\kappa^2 - \sigma) = 0.$$

Therefore either κ satisfies (7) or κ is constant. For an arc length parametrized curve the determinant $[x', x'']$ is curvature. Here we have

$$[x', x''] = \frac{4}{(\kappa_m^2 - \sigma)^2}(2(\kappa')^2\kappa - \kappa''(\kappa^2 - \sigma)). \tag{19}$$

If κ is constant the curvature of x is zero, hence x is a trivial elastic curve. If κ satisfies (7) we substitute in (19) κ'' according to (7) and κ'^2 according to (18). This yields

$$[x', x''] = \kappa.$$

$\square$

The representation formula (17) of an elastic curve is extremely useful for the computation of interpolating elastica. One reason for this is that (17) involves no trigonometric functions and fewer integrations than the standard representation based on the formula $x' = (\cos\Psi, \sin\Psi)$. Therefore (17) is less expensive to evaluate. Furthermore (17) can be used to find piecewise polynomial approximations of elastic curves based on a spline approximation of the curvature function. The author has established polynomial splines which approximate the curvature functions of plane elastica with high accuracy. Using these piecewise polynomial curvature functions together with the new representation (17) one obtains polynomial spline approximations of the elastica itself. These approximations will be discussed in [4].

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